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Space saving calculation of symbolic resultants

Jane M Pearson and Noel G Lloyd

Abstract. We describe an approach to the computation of symbolic resultants in which factors are removed during the course of the calculation, so reducing the stack size required for intermediate expressions and the storage space needed. We apply the technique to three well-established methods for calculating resultants. We demonstrate the advantages of our approach when the resultants are large and show that some otherwise intractable problems can be resolved. In certain cases a significant reduction in the cpu time required to calculate the resultant is also evident.

Keywords. symbolic computation, multivariate resultants.

1. Introduction

Our interest in the calculation of symbolic resultants arose from our research into some of the properties of systems of nonlinear differential equations (see for example [9],[10],[11],[12]). In order to provide a context for our discussion we first give a brief description of the mathematical background. We consider differential systems of the form

$$\dot{x} = \lambda x + y + p(x, y), \quad \dot{y} = -x + \lambda y + q(x, y), \tag{1}$$

where p and q are polynomials without linear or constant terms. The corresponding complex form of system (1) is

$$i\dot{z} = (1 + i\lambda)z + \sum_{i+j=2} A_{ij} z^i \bar{z}^j, \tag{2}$$

where $z = x + iy$, $A_{ij} \in \mathbb{C}$. When $\lambda = 0$ the origin is said to be a *fine focus*. Our objective is to establish the conditions under which the origin is a centre and to determine the maximum number of limit cycles that can be bifurcated from the origin for systems of the form (1) or (2) under perturbation of the coefficients. **All** orbits in the neighbourhood of the origin when it is a *centre* are closed; in contrast a *limit cycle* is an isolated closed orbit.

We separate the calculation of the conditions for the origin to be a centre into two parts: necessity and sufficiency. It is in the calculation of the necessary conditions that most of the large resultant calculations arise. These conditions are obtained by calculating the *focal values*, which are polynomials in the coefficients in p and q, (or in the A_{ij}) and are defined as follows. There is a function V, analytic in a neighbourhood of the origin, such that the rate of change along orbits, $\dot{V} = \eta_2 r^2 + \eta_4 r^4 + \cdots$, where $r^2 = x^2 + y^2$ and $\eta_2 = \lambda$. The η_{2k} are the focal values and the number of terms in each η_{2k} grows rapidly as k increases. In examples where η_4 has only four terms it is not unusual for η_{12} to have over a thousand terms, and η_{14} over three thousand.

The origin is a centre if, and only if, all the focal values are zero. By the Hilbert basis theorem, the ideal they generate has a finite basis so there is M such that if $\eta_{2j} = 0$, for $j \leq M$, then $\eta_{2j} = 0$ for all j. The value of M is not known *a priori*. The origin is a fine focus of *order* k if $\eta_{2m} = 0$, for $m \leq k$ but $\eta_{2k+2} \neq 0$. At most k small amplitude limit cycles can bifurcate from a fine focus of order k.

Our approach is to calculate the first few focal values for a given system and to make substitutions from each one into the other focal values. We have $\eta_2 = \lambda = 0$, necessarily. We make a substitution from $\eta_4 = 0$, for one of the variables, into subsequent calculated focal values. Then we make substitutions from $\eta_6 = 0$, $\eta_8 = 0$ and so on. We thus obtain expressions for each of the eliminated variables in terms of the remaining variables - information that is required for the bifurcation of the limit cycles. After each substitution we remove common factors from the remaining calculated focal values, these being candidates for the conditions under which the origin is a centre. We continue until we can show that if the remaining factor of focal value η_{2k} is zero then focal value η_{2k+2} is necessarily non-zero. Then the maximum order of the origin as a fine focus is k. The sufficiency of the candidate centre conditions is proved independently using a range of different techniques.

We have considered looking for a Gröbner basis for the set of focal values; there are three main drawbacks to this approach. First, we do not know *a priori* the value of M for a given differential system. Secondly, the Gröbner basis does not readily give us the information we require in order to bifurcate the limit cycles. Finally, obtaining the Gröbner basis is non-trivial for many systems.

In the systems of interest to us the focal values usually involve at least seven variables. As each variable is eliminated the remaining focal values grow; they contain more terms, the variables occur to higher degrees and the integer coefficients become larger. At each stage of the elimination process an attempt is made to simplify the focal values by factorising them, each such factor is then considered individually. However we inevitably reach a point where the variable we wish to eliminate does not occur linearly in any of the focal values (or factors of the focal values) and we must employ polynomial remainder sequences, as in [10],[12] or use resultant calculations, see for example [12], to eliminate that variable. Often successive resultant calculations are required; the performance of such calculations is sensitive to the order in which the variables are eliminated.

Typically we find the variable being eliminated occurs to degree greater than 10, with the other variables occurring to higher degrees. The integer coefficients often have as many digits as the total degree of the polynomial. Procedures for calculating resultants fail for some examples simply because the expressions involved become so massive. Techniques to make the computations more feasible are essential. In this paper we describe how we have developed software to calculate the resultant of multivariate polynomials which exploits the particular features we have observed in previous examples. Many of the resultants we wish to calculate have several simple factors occurring to high multiplicity; we aim to remove as many of these factors as we can during the course of the calculation, so reducing the size of intermediate expressions. Some of these factors, such as common factors of leading coefficients of the variable being eliminated, can be predicted. Others are determined by factorisations or greatest common divisor (gcd) calculations.

The resultant can be calculated in various ways; we consider three approaches, each of which requires the evaluation of the determinant of a certain matrix. The Sylvester, Bézout and Companion matrices which we use are described in section 4. The most time and space consuming element of any of these approaches is the calculation of the determinant, the setting up of the matrix always being a minor consideration. In section 5 we present a method for calculating the determinant in which intermediate expression swell is reduced by removing factors of the resultant as they arise. This idea can be applied to any method that involves the calculation of a determinant.

There are techniques to compute the resultant of three polynomials with respect to two variables directly, using for example the Macaulay determinant [14],[15] or Dixon resultants [8]. The Dixon resultant is a generalisation of the Bézout-Cayley method which is described below. A major limitation of the Dixon based resultants is that often an extraneous factor is generated, a problem addressed in [2]. The size of the matrix is dependent on the variable ordering and it is essential to minimize this to enable the effective calculation of the determinant. The size of the Macaulay matrix is given by $\binom{d-2}{2}$, where d is the sum of the total degree of the two variables being eliminated in each of the three polynomials. In the examples we encounter d is likely to be at least 50. Although we do not consider this approach here it would be possible to apply the idea of the early removal of factors to these resultant calculations.

We demonstrate our approach by reference to two examples which come from the investigation of two systems which we describe in section 3. Section 2 contains a general discussion of resultant calculations. In section 6 we present two further examples which highlight the advantages of the Bézout or Companion matrix approaches. Our concluding remarks are in section 7.

2. Resultants

Typically we have two multivariate polynomials, say f, g, each with several hundred, and often several thousand, terms and we wish to establish under what circumstances $f = g = 0$. We denote the resultant of f and g with respect to the variable x by $res(f, g, x)$. We have $f = g = 0$ only if $res(f, g, x) = 0$. To simplify further calculations we require all the irreducible factors of $res(f, g, x)$.

The computation of symbolic resultants of large multivariate polynomials (for example polynomials of degree greater than ten in the variable being eliminated with coefficients that are polynomials in one, or more, variables occurring to degrees greater than ten) is very demanding of both computer space and time. It is often impossible to obtain the resultants we want using the currently available software. Of the techniques available most involve the calculation of the determinant of a matrix, the exact form of the matrix reflecting the different methods. It is also possible to calculate resultants using interpolation techniques [13]. First the degree of the required resultant is determined, then resultant calculations are performed for specific values of one of the variables. Finally a polynomial is interpolated from this data set. For this to be effective one requires an efficient means of calculating the resultants and an efficient method for interpolating polynomials in several variables.

We note that in many of our examples the resultant contains many simple, repeated factors and we exploit this by removing any such factors as they arise during the calculation of the determinant. In this paper we concentrate on the calculation of the resultant with respect to one variable but our approach could easily be applied to any method which requires a determinant to be computed. Our computations were performed on a Compaq Alpha XP1000 workstation, with single 667MHz Alpha EV5 processor and 1 Gb of memory, using the Computer Algebra systems REDUCE and Maple.

The resultant of two given polynomials with respect to a given variable can be thought of as the elimination of the given variable from the two polynomials. Where the polynomials have a non-trivial gcd their resultant vanishes. Writing the irreducible, multivariate polynomials f and g as polynomials in the single variable x, with polynomial coefficients in the remaining variables, we have

$$f = \sum_{i=0}^{n} a_i x^i \text{ and } g = \sum_{i=0}^{m} b_i x^i, \tag{3}$$

where $m \geq n$. Our requirement is to find all factors of $res(f, g, x)$. We present a technique for identifying some of these factors during the course of the calculation of the resultant. Removing such factors during the computation reduces the size of the intermediate expressions calculated and consequently makes the overall calculation that much more feasible.

3. Examples

We compare three different methods, based on the Sylvester, Bézout and Companion matrices respectively, by reference to examples. The first two examples are straightforward, we include them merely to demonstrate the differences in the three methods that we consider. In particular we are interested in when factors of the resultant first occur in the calculation. Two further examples are presented in section 6, these illustrate the effectiveness of our technique.

Example 1 The first example arose in the investigation of cubic differential systems of the form

$$i\dot{z} = z + A_{11}z\bar{z} + A_{30}z^3 + A_{12}z\bar{z}^2 + A_{03}\bar{z}^3, \tag{4}$$

where $A_{ij} \in \mathbb{C}$. At a certain point in the elimination of variables from the focal values for this system we require $R_0 + aR_1 = 0$, where $R_0(b, c, d), R_1(b, c, d)$ are real non-homogeneous polynomials of degrees $5, 4$ respectively in d, and a, b, c, d are functions of the real and imaginary parts of the A_{ij}. We need to consider the two possibilities $R_1 \neq 0, a = -\frac{R_0}{R_1}$ and $R_0 = R_1 = 0$. Here we are concerned with the second case and we calculate the resultant of R_0 and $R_1$with respect to d . We refer to this as Example 1.

Example 2 The second example arose in the investigation of differential systems of the form

$$\begin{aligned} \dot{x} &= y(1 + kx), \\ \dot{y} &= -x + c_1x^2 + c_2xy + c_3y^2 + c_4x^3 + c_5x^2y + c_6xy^2 + c_7y^3, \end{aligned} \tag{5}$$

when (5) has two coexisting fine foci. The origin is a fine focus and we can scale the system so that there is a second fine focus at $(1, 0)$; hence $c_4 = 1-c_1$, $c_5 = -c_2$ and $(k+1)(c_1-2) > 0$. The variables k, c_6, c_7 are eliminated from the focal values leaving polynomials in the remaining variables c_1, c_2, c_3. Further details can be found in [7]. In particular we have two polynomials S_0, S_1 of degrees 6,4 in c_2^2 with coefficients that are polynomials in c_1 and c_3. We refer to the resultant of S_0 and $S_1$with respect to c_2^2 as Example 2.

4. The matrices

Considering the polynomials in (3) we first define the form of the matrices used in the three approaches to the calculation of resultants and note some of their properties.

4.1. Sylvester matrix

The Sylvester matrix, S, is defined as

$$S = \begin{array}{cc} \begin{pmatrix} b_m & b_{m-1} & . & . & . & b_0 & 0 & . & . & 0 \\ 0 & b_m & b_{m-1} & . & . & b_1 & b_0 & 0 & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & 0 & b_m & b_{m-1} & . & . & . & b_0 \\ a_n & a_{n-1} & . & . & a_0 & 0 & . & . & . & 0 \\ 0 & a_n & a_{n-1} & . & . & a_0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & a_n & . & . & . & a_0 \end{pmatrix} & \begin{array}{c} \text{row } 1 \\ . \\ . \\ . \\ n \\ n+1 \\ . \\ . \\ . \\ n+m \end{array} \end{array}$$

where the first n rows of the matrix contain only the coefficients of the polynomial g, followed by m rows with only the coefficients of f. The Sylvester matrix is generally large, its order being the sum of the degrees of the polynomials f and g, in x, but it is sparse in that at most $2mn+m+n$ of its elements are non-zero. The Sylvester matrix need not be stored explicitly for our procedure to determine the resultant, since we require only the individual coefficients a_i and b_i. The resultant of f and g is given by

$$res(f, g, x) = (-1)^{mn} \det(S);$$

as is explained in [16] and discussed further in [3]. The sign of the resultant is not important for our purposes as we are only interested in the conditions under which the resultant is zero. Clearly common factors of the coefficients occurring in any column of the Sylvester matrix are factors of the resultant. In practice we find that, in our problems, only columns 1 and $m+n$ are likely to have such factors.

4.2. Bézout matrix

The approach to finding a resultant using the Bézout matrix is often known as Cayley's method [1],[6]. Let

$$\beta(x, y) = \frac{f(x)g(y) - f(y)g(x)}{(x-y)}.$$

Clearly $(x-y)$ is a factor of the numerator, so β is a polynomial of degree $m-1$ in x and y. The elements of the Bézout matrix , B, are given by

$$B(i, j) = \text{coefficient of } x^{i-1}y^{j-1} \text{ in } \beta$$

for $i, j = 1, 2, \ldots, m$. The resultant in this case is given by

$$res(f, g, x) = \pm b_m^{n-m} \det(B),$$

where $b_m \neq 0$. The $m \times m$ Bézout matrix is symmetric, since $\beta(x, y) = \beta(y, x)$, so it is only necessary to store the upper (or lower) triangle of B. We consider those elements in the upper triangle, which has the following form:

$$\begin{pmatrix}
B_{1,1} & B_{1,2} & . & . & . & . & . & . & . & B_{1,m} \\
 & B_{2,2} & . & . & . & . & . & . & . & B_{2,m} \\
 & & . & . & . & . & . & . & . & . \\
 & & & . & . & . & . & . & . & . \\
 & & & & B_{n+1,n+1} & . & . & . & . & B_{n+1,m} \\
 & & & & & B_{n+2,n+2} & . & . & B_{n+2,m-1} & 0 \\
 & & & & & & . & . & 0 & 0 \\
 & & & & & & & . & . & . \\
 & & & & & & & & & 0
\end{pmatrix}$$

The entries can be determined according to a procedure described in [5] suitably modified for $m \neq n$. Let $ab_{i,j} = a_j b_{i-1} - a_{i-1} b_j$, with $a_p = 0$ if $p > n$. Then $B(1,j) = ab_{1,j}$ for $j = 1, \dots, m$, $B(i,m) = ab_{i,m}$ for $i = 2, \dots, \min(m, n+1)$, and $B(i,j) = ab_{i,j} + B(i-1, j+1)$ for $i = 2, \dots, m$, $j = i, \dots, m-1$. The number of elements to be stored is $\tau = \frac{1}{2}m(m+1)$ when $m = n$, $\tau - \frac{1}{4}(m-n-1)(m-n+1)$ when $m - n > 0$ is odd and $\tau - \frac{1}{4}(m-n-2)(m-n+2) - 1$ when $m - n \geq 2$ is even.

The expression for $det(B)$ differs from the resultant being sought by a factor of b_m^{m-n}, if $m \neq n$. We must remove at least the $m-n$ factors b_m during the course of the resultant calculation to avoid creating an expression which is even larger than the required resultant.

Lemma 1. *If $m - n = 1$ then b_m is a factor of row, or column, m. If $m - n > 1$ then b_m is a factor of row and column m.*

Proof. We have $B(i,m) = B(m,i) = -a_{i-1} b_m, i = 1, \dots, n+1$ and if $m - n > 1$ then $B(i,m) = B(m,i) = 0, i = n+2, \dots, m$. The proof follows. $\square$

We note that as we are working with only the upper triangle of B the factor b_m can be removed from row/column m only if $m - n > 1$.

4.3. Companion matrix

The Companion matrix for the polynomial f is defined as

$$C_f = \begin{pmatrix}
0 & . & . & . & -a_0/a_n \\
1 & 0 & . & . & -a_1/a_n \\
0 & 1 & 0 & . & -a_2/a_n \\
. & . & . & . & . \\
0 & . & . & 1 & -a_{n-1}/a_n
\end{pmatrix}$$

where the leading coefficient of f is strictly non-zero. We note that C_f is an $n \times n$ matrix, the lowest ordered matrix of the three being considered, and we have, as described in [4], that

$$res(f, g, x) = \pm a_n^m \det(g(C_f)).$$

The form of the matrix C_f is such that C_f^2 is the matrix C_f with column one removed and column n with entries $A_{1,1}/a_n^2,\ A_{1,2}/a_n^2, \dots, A_{1,n}/a_n^2$, where $A_{1,1} =$

$a_0 a_{n-1}$ and $A_{1,i} = a_{i-1}a_{n-1} - a_{i-2}a_n$, for $i = 2, \ldots, n$. In general C_f^p is C_f^{p-1} with column one removed and a new column n with entries $A_{p-1,i}/a_n^p$, where $A_{0,i} = -a_{i-1}$, $A_{k,1} = -a_0 A_{k-1,n}$, $A_{k,i} = a_n A_{k-1,i-1} - a_{i-1}A_{k-1,n}$ for $k = 2, \ldots, m-1$ and $i = 1, \ldots, n$. Hence the elements of the matrix $G = g(C_f)$ are given by

$$G(i,j) = \Big(\sum_{t=n-j+1}^{m} A_{t-n+j-1,i} b_t a_n^{m-t} + \Omega\Big)/a_n^{m-n+j}, \tag{6}$$

where $\Omega = 0$ if $i < j$ and $\Omega = b_{i-j}a_n^{m-n+j}$ if $i \geq j$, for $i, j = 1, \ldots, n$. The matrix G is dense. We note that the elements of column j of the matrix G have a common factor of $a_n^{-(m-n+j)}$. Writing C as the matrix G with these common factors removed we have

$$res(f,g,x) = \pm \prod_{i=m-n+1}^{m-1} a_n^{-i} \det(C) = \pm a_n^{(n-1)(\frac{1}{2}n-m)} \det(C).$$

We refer to C as the reduced Companion matrix for f and g.

Lemma 2. *The elements of C have a common factor that is the greatest common divisor of a_n and b_m.*

Proof. From (6) every element of C has a_n as a factor unless $t = m$ in which case b_m is a factor. □

As $m \geq n$ the determinant we need to calculate is a larger expression than the required resultant, therefore it is essential to develop a technique in which the factors a_n are removed as they arise.

5. The determinant

The methods described above require the evaluation of the determinant of a symbolic matrix to obtain the resultant. There are efficient methods for calculating the determinant of a numerical matrix and most can be applied to matrices with symbolic entries. However the time taken when the matrix contains symbolic elements can be significantly longer. This is partly because mathematical operations on symbolic expressions are performed by software rather than hardware.

Suppose that $M = (M_{ij})$ is an $s \times s$ matrix. The determinant of M is usually defined recursively, as the sum of certain matrix elements times the determinants of matrices that are the cofactors of those elements. Although conceptually simple, computationally this is not a good definition to use as the time taken to evaluate the determinant of a matrix M grows exponentially as s increases. We adopt a direct approach in which we start by evaluating the determinants of 2×2 matrices and progress through the determinants of 3×3 matrices until we eventually calculate $\det(M)$. This allows us to remove common factors of the sub-determinants at each stage of the process.

Let $H_k(i_1, i_2, \ldots, i_k)$ be the $k \times k$ sub-determinant formed from rows $i_1, \ldots, i_k$ of the first k columns of M, with $H_1(i) = -M_{i,1}$. We define iteratively

$$H_k(i_1, i_2, \ldots, i_k) = \sum_{r=1}^{k} (-1)^{r-1} M_{i_r,k} H_{k-1}(i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_k) \qquad (7)$$

for $k = 2, \ldots, s$. Then $\det(M) = H_s(1, \ldots, s)$. For each k there are potentially $\binom{s}{k}$ sub-determinants to be calculated.

Lemma 3. *Any common factor of the $k \times k$ sub-determinants is also a factor of* $\det(M)$.

Proof. If α is a common factor of the H_k for $k = \kappa$ then by (7) it is also a factor of the $H_{\kappa+1}$. It follows that α is a factor of $\det(M) = H_s$. □

The approach we adopt is to remove common factors from the H_k, before calculating the H_{k+1}. In general the number and size of the expressions for the H_k is such that calculating greatest common divisors (or factorizing each one) at each level is non-trivial. So we remove any common factors that we know will be present, for certain H_k, before attempting a gcd calculation. For example, in the Bézout and reduced Companion matrices we know that the leading coefficient b_m and multiplicities of a_n respectively must be present as factors in certain of the H_k. In the Sylvester matrix we know that for $k < n$ only common factors of a_n and b_m can arise in the H_k. Factors of the leading coefficients occur repeatedly in the resultant, so we check for them being common factors in the H_k even when we do not explicitly predict their presence. Integer factors of the H_k are easily lifted and a greatest common divisor for them can be readily computed. Removing this common factor in itself stems the growth of the integer coefficients. Other common factors that may occur depend very much on the problem and our objective is to develop a strategy of looking for such factors at the earliest opportunity.

5.1. Determinant of the Sylvester matrix

The calculation of the determinant of the Sylvester matrix using our method gives rise to some unique problems due to the sparsity of the matrix. It is desirable not to waste time calculating any H_k that is known *a priori* to be zero and, more importantly, we do not wish to calculate non-zero H_k that are not required in the determination of H_{k+1}. We have $H_1(1) = -b_m, H_1(n+1) = -a_n$ and all other H_1 are zero. There are five non-zero H_2, namely

$$\begin{aligned} H_2(1,2) = {} & b_m^2, \quad H_2(1,n+1) = {} b_m a_{n-1} - b_{m-1} a_n, \\ H_2(1,n+2) = {} & b_m a_n, \ H_2(2,n+1) = {} -H_2(1,n+2), \ H_2(n+1,n+2) = a_3^2. \end{aligned}$$

When $k \le n$, each $k \times k$ sub-matrix can notionally be reduced to an upper triangular matrix, the determinant of which is zero if there is a zero entry on the diagonal. Referring to (7) we say that $i_1, i_2, \ldots, i_k$ are the parameters of H_k. Let $h_{k,i} = \{i_1, i_2, \ldots, i_k\}$ for each H_k, where $i = 1,\ldots$,number of H_k. The determinants we must calculate for $k = \kappa \le n$ are as follows:

- the H_κ with parameters $h_{\kappa-1,i}$ plus either κ or $n+\kappa$,
- the H_κ for matrices formed from κ of the rows occupied in column $\kappa-1$. We define a row to be of type r if the first entry in the row is in column r. The rows chosen must be of types 1 to κ, two selected rows of type ρ generate a row of type $\rho+1$. The selection of κ rows from the possible $2(\kappa-1)$ rows made up of two rows of each type r, for $r=1,\dots,\kappa-1$, will include at least two rows of the same type. We refer to the selection of two rows of the same type, that is rows i and $n+i$, as an 'and', as opposed to an 'or' when row i or row $n+i$ is chosen. We consider all possible combinations of 'and's and 'or's - if p 'and's are included then $\kappa-2p$ 'or's are required, for $p=1,\dots,\lfloor\frac{\kappa}{2}\rfloor$. We ensure that our selection results in one row of each of the required types, otherwise the sub-determinant is zero. For example, row 1 or row $n+1$, or both must be chosen and no $h_{\kappa,i}$ can involve rows $\kappa-1$, $n+\kappa-1$, $\kappa-2$, and $n+\kappa-2$ simultaneously.

For $k=n+1,\dots,m-1$ we could modify the technique used for $k\le n$; for $k=\kappa$ add $n+\kappa$ to each $h_{\kappa,i}$ and consider combinations of rows occupied in column $\kappa-1$. However this leads to a small number of sub-determinants being calculated that are not required. We adopt an alternative approach of recursively deleting from the integers $Z=\{1,2,\dots,m+n\}$ the occupied rows of columns $m+n, m+n-1,\dots,\kappa+1$ to determine the $h_{\kappa,i}$. Starting with Z we first delete either n or $m+n$, then from $Z-\{n\}$ we delete $n-1$ or $m+n-1$ or $m+n$ and from $Z-\{m+n\}$ we delete $n-1$ or $m+n-1$, and so on. A few of the sub-determinants calculated will be zero but this is considered preferable to calculating non-zero quantities that are not needed.

The form of the Sylvester matrix means that we can determine which H_k are required for $k=m,\dots,m+n-1$ from a knowledge of which H_j, for $j=1,\dots,n$ are to be calculated. For each of the parameters in $h_{j,i}=\{i_1,i_2,\dots,i_j\}$, if $i_r\le n$ then $j_r=m+n-i_r+1$ else $j_r=2n-i_r+1$ is deleted from the integers $1,2,\dots,m+n$ to give $h_{m+n-j,i}$. For example, assuming that $n\ge 2$, the five non-zero H_{m+n-2} are

$$\begin{aligned}H_{m+n-2}(1,\dots,m+n-2), H_{m+n-2}(1,\dots,n-2,n+1,\dots,m+n),\\ H_{m+n-2}(1,\dots,n-2,n,\dots,m+n-1),\\ H_{m+n-2}(1,\dots,n-1,n+1,\dots,m+n-1),\\ H_{m+n-2}(1,\dots,n-1,n+1,\dots,m+n-2,m+n).\end{aligned}$$

The number of sub-determinants which must be calculated rises steeply as k increases, reaching a maximum when $k=\lfloor\frac{m+n}{2}\rfloor$. We have already seen that five sub-determinants are required when $k=2$. The following table shows the number required for values of $k\le 10\le n$. The increase in number as k goes from $n+1$ to $\lfloor\frac{m+n}{2}\rfloor$ is not as rapid and for every value of k significantly fewer sub-determinants are required than would be the case for a dense matrix.

k	2	3	4	5	6	7	8	9	10
no. of H_k	5	14	42	132	429	1430	4862	16796	58786

Having decided which of the sub-determinants are to be calculated we turn to the question of common factors and their occurrence.

Lemma 4. *Common factors of the sub-determinants H_k , for $k \leq n$, for the Sylvester matrix are common factors of the leading coefficients of the polynomials f, g. Common factors of the H_k, for $n+1 < k \leq m$ are factors of the leading coefficient of f.*

Proof. We have that when $k \leq n$, $H_k(1, 2, \ldots, k) = b_m^k$ and when $k \leq m$, $H_k(n+1, n+2, \ldots, n+k) = a_n^k$. The proof follows. □

We cannot expect to find any other common factors of the H_k for $k < m+1$, so we defer the calculation of greatest common divisors until we reach values of $k \geq m+1$. At this point the number of sub-determinants for each k is decreasing, although the expressions generated for each one are large.

Sylvester algorithm

Begin
Input f, g
$H_1(1) = b_m/gcd(a_n, b_m)$
$H_1(n+1) = a_n/gcd(a_n, b_m)$
$H_2(1, 2) = H_1(1)b_m$
$H_2(1, n+1) = H_1(1)a_{n-1} - b_{m-1}H_1(n+1)$
$H_2(1, n+2) = H_1(n+1)b_m$
$H_2(n+1, n+2) = H_1(n+1)a_n$
Remove factors of the $gcd(a_n, b_m)$ *that are common to the* H_2
$H_2(2, n+1) = -H_2(1, n+2)$
$k = 3$ *step* 1 *until* n
    Determine the $h_{k,i}$ *for each* H_k *to be calculated*
        Add k *or* $n+k$ *to* $h_{k-1,i}$
        Select k *occupied rows of column* $k-1$, *of different row types*
    Calculate the H_k
    Remove common numerical factor from the H_k
    Remove factors of a_n *that are common to the* H_k
Z=$\{$*1,2,...,m+n*$\}$
$k = n+1$ *step* 1 *until* $m-1$
    Determine the $h_{k,i}$ *for each* H_k *to be calculated*
        Delete from Z the occupied rows of columns $m+n, m+n-1, \ldots, k+1$
    Calculate the H_k
    Remove common numerical factor from the H_k
    Remove factors of a_n *that are common to the* H_k
$k = m$ *step* 1 *until* $m+n-1$
    Determine the $h_{k,i}$ *for each* H_k *to be calculated*
        $r = 1$ *step* 1 *until* $m+n-k$
            $i_r = r^{th}$ *element of* $h_{m+n-k,i}$
            If $i_r \leq n$ *then delete* $m+n-i_r+1$ *else* $2n-i_r+1$ *from Z*

Calculate the H_k
Remove common numerical factor from the H_k
Remove factors of a_n *that are common to the* H_k
If $k > m$
Find and remove common factors of the H_k
Calculate H_{m+n}
Return H_{m+n}*, list of factors removed together with their multiplicities*
End of algorithm

We illustrate this approach using the two examples described in section 3.

Example 1a We use our technique based on sub-determinants of the Sylvester matrix to calculate the resultant of the polynomials of Example 1, namely R_0 and R_1, with respect to d. The degrees of d in R_0, R_1 are $5, 4$ respectively, so the resultant is given by H_9. The leading coefficients of d in R_0, R_1 have the factor $75(b+3)(3b-1)$ in common and $(162b+5)\varphi_4(b)$, where φ_4 is a polynomial of degree 4 in b, is a factor of the leading coefficient in R_1. We find

$$res(R_0, R_1, d) = 16A(162b+5)^2\gamma^3\varphi_4^2(b)\varphi_3^2(b)\Psi_1(b,c)\Psi_2(b,c), \tag{8}$$

with $\gamma = (3b^2c+12b^2+8bc+32b-3c-27)$, $A = 1080000(3b-1)^3(b+3)(162b+5)\gamma$, ψ_3 is a polynomial of degree 3 in b, Ψ_1 is a polynomial of degrees $8, 2$ in b, c respectively and Ψ_2 is degree 40 in b, 8 in c. Here A is removed from the resultant in the form of common factors of the H_k during the course of the procedure and the other factors are determined by factorisation of H_9. We remove $75(b+3)(3b-1)$ from the H_1, then $15(3b-1)$ is common to the H_2 and the H_3. Subsequently only common numerical factors are encountered until the H_8 are calculated, when $(162b+5)\gamma$ is found.

Example 2a In Example 2 the polynomials S_0, S_1 are of degrees 6,4 in c_2^2 with coefficients that are polynomials in c_1 and c_3. We calculate the resultant

$$res(S_0, S_1, c_2^2) = 11520A(c_1+c_3-1)^3(3c_1+3c_3-2)^2\Omega(c_1,c_3), \tag{9}$$

where $\Omega(c_1, c_3)$ is of degree 44 in c_1, c_3 and $A = 4976640(c_1+c_3-1)^5\delta$, with $\delta = (15c_1^2+30c_1c_3-20c_1+15c_3^2-20c_3+8)$, is removed from the resultant in the form of common factors of the H_k during the course of the procedure. The other factors are obtained by factorising H_{10}. Here the common factor of the leading coefficients $10(c_1+c_3-1)\delta$ is removed from the H_1, then $2(c_1+c_3-1)$ is common to the H_2. Again only numerical common factors are found until the H_8, which have $72(c_1+c_3-1)$ in common. The remaining factor $288(c_1+c_3-1)^2$ of A is found in the H_9.

5.2. Determinant of the Bézout matrix

When applying our method of sub-determinant calculations to the Bézout matrix it is advantageous to re-order the columns so that any zero elements occur in the first few columns. This ensures that the H_k for $k \le m-n-1$ are as simple as possible. Now the $H_2(i,j)$ are non-zero for $i = 1, 2, \ldots, n+1 \le m-1$, $j = 2, \ldots, n+2 \le m$ and in general we must evaluate $\binom{\min(m,n+k)}{k}$ sub-determinants for each $k = 2, \ldots, m$. Symmetry allows us to store only the upper triangle of the Bézout matrix, but we must adjust our calculation of the H_k accordingly. By Lemma 1 if $m-n > 1$ a factor b_m^2 can be removed from the matrix, the $m-n-2$ factors b_m that remain to be removed to give the final expression for the determinant occur in the H_k for $k = 2, \ldots, m-n-1$. When $m-n = 1$ then b_m is a factor of the H_2. Although the number of sub-determinant calculations required is considerably fewer than in the calculation of the determinant of the Sylvester matrix each individual sub-determinant is more complicated.

Bézout algorithm

Begin
Input f,g
Store factors of b_m and their multiplicities
Set up upper triangular $m \times m$ Bézout matrix
$i = 1$ *step* 1 *until* $min(m, n+1)$
    $j = i$ *step* 1 *until* n
        $B(i,j) = b_{i-1}a_j - b_j a_{i-1}$
    if $m > n$ *then*
        $j = n+1$ *step* 1 *until* m
            $B(i,j) = -a_{i-1}b_j$
$i = 2$ *step* 1 *until* $min(m-1, n+1)$
    $j = i$ *step* 1 *until* $m-1$
        $B(i,j) = B(i,j) + B(i-1, j+1)$
$i = n+2$ *step* 1 *until* $(m+n+1)/2$
    $j = i$ *step* 1 *until* $m+n-i+1$
        $B(i,j) = B(i-1, j+1)$
If $m - n > 1$ *remove common factor* b_m *from column* m
Calculate the $\binom{min(m,n+2)}{2}$ *sub-determinants* H_2
$i = 1$ *step* 1 *until* $m-2$
    $j = i+1$ *step* 1 *until* $m-1$
        $H_2(i,j) = B(i, m-1)B(j,m) - B(i,m)B(j, m-1)$
$i = 1$ *step* 1 *until* $m-1$
    $H_2(i,m) = B(i, m-1)B(m,m) - B(i,m)B(m-1, m)$
Find and remove common numerical factor of the H_2
Remove common non-numerical factor of b_m *from the* H_2
$k = 3$ *step* 1 *until* $m-1$
    Combine column $m-k+1$ *and the* H_{k-1} *to give the* $\binom{n}{k} H_k$
    Remove common numerical factor from the H_k
    If $k < m-n-1$
        Remove non-numerical factors of b_m *that are common to the* H_k
    Else
        Find and remove common factors of the H_k
Calculate H_m
Return H_m*, list of factors removed together with their multiplicities*
End of algorithm

Example 1b We calculate the resultant for the polynomials given in Example 1 using our technique applied to the Bézout matrix. Here $m = 5, n = 4$, so we cannot remove b_5 from the stored matrix elements. Common factors of the H_k can occur for any value of k. Again we obtain the resultant as given by (8) with A removed during the course of the procedure. The common factor of the H_2 is $150b_5$, where $b_5 = 225(3b-1)(b+3)\varphi_5(b)$ and $\varphi_5(b)$ is a polynomial of degree 5 in b. We have

$60(3b-1)^2(b+3)$ common to the H_3 and $120(3b-1)(162b+5)\gamma$ common to the H_4. The remainder of the determinant is given by the factorisation of H_5.

Example 2b Similarly we calculate the resultant for Example 2 and obtain the result given by (9). In this case b_6^2 is removed from the matrix before the determinant is calculated, hence the resultant is equal to H_6. We find $10(c_1+c_3-1)\delta$ is common to the H_2, $24(c_1+c_3-1)$ to the H_3, $72(c_1+c_3-1)$ to the H_4 and finally $288(c_1+c_3-1)^2$ to the H_5.

5.3. Determinant of the Companion matrix

We apply our method of sub-determinant calculations to the approach based on the Companion matrix. The number of sub-determinants H_k that must be calculated is $\binom{n}{k}$ for $k=2,\ldots,n-1$. We note that although the matrix is dense fewer sub-determinants are required than for the Bézout or Sylvester matrices. The entries in the reduced Companion matrix are even more complex than those of the Bézout matrix.

Lemma 5. *The sub-determinants H_k, for fixed $k>2$, for the reduced Companion matrix for f and g have a common factor $a_n^{m-n+k-1}$ after a factor $a_n^{m-n+k-2}$ has been removed from H_{k-1}.*

Proof. Consider first the H_2. We have for $p=1,2,\ldots,n-1$ and $s=p+1,\ldots,n$

$$\begin{aligned} H_2(p,s) &= C(p,1)C(s,2)-C(s,1)C(p,2) \\ &= (\sum_{t=n}^{m} A_{t-n,p}b_t a_n^{m-t}+b_{p-1}a_n^{m-n+1})(\sum_{t=n-1}^{m} A_{t-n+1,s}b_t a_n^{m-t}+\Omega_s) \\ &\quad -(\sum_{t=n}^{m} A_{t-n,s}b_t a_n^{m-t}+b_{s-1}a_n^{m-n+1})(\sum_{t=n-1}^{m} A_{t-n+1,p}b_t a_n^{m-t}+\Omega_p) \end{aligned}$$

where $\Omega_v=0$, if $v<2$ and $\Omega_v=b_{v-2}a_n^{m-n+2}$ otherwise. The coefficient of the term in b_m^2 is

$$\begin{aligned} & A_{m-n,p}A_{m-n+1,s}-A_{m-n,s}A_{m-n+1,p} \\ =\ & (a_nA_{m-n-1,p-1}-a_{p-1}A_{m-n-1,n})(a_nA_{m-n,s-1}-a_{s-1}A_{m-n,n}) \\ & -(a_nA_{m-n-1,s-1}-a_{s-1}A_{m-n-1,n})(a_nA_{m-n,p-1}-a_{p-1}A_{m-n,n}). \end{aligned}$$

Clearly a_n is a factor; repeated application of the recurrence relation for the As gives a_n^{m-n+1}as a factor of the coefficient of b_m^2. Similarly for all the coefficients of b_ib_j. We conclude that a_n^{m-n+1} is a factor of the H_2 ,which is removed.
In general the H_k involve the entries of column k of the matrix C, for which the summations are over $t=n-k+1,\ldots,m$. Terms in H_k independent of a_n cancel and repeated application of the recurrence relation for the As gives $a_n^{m-n+k-1}$ as a factor. □

According to Lemma 2 the gcd of a_n, b_m is a factor of each element of the matrix C, we remove this factor before calculating the sub-determinants. We also remove any numerical factors that are common to the rows or columns of C. The common factors of the H_k, that we know will arise according to Lemma 5, are removed for each k. The result given in Lemma 5 means that H_n is divisible by a_n^{m-1}, and unless common factors of the H_k, for $k < n$, ultimately reduce H_n considerably the expression generated could be larger than that for the resultant.

Companion algorithm

Begin
Input f, g
Calculate $\gcd(a_n, b_m)$
Set up $n \times m$ *matrix* A
j=1 step 1 until n
  $A(0, j) = -a_{j-1}$
$A(1, 1) = A(0, 1)A(0, n)$
i=2 step 1 until m-1
  A(i,1)=A(0,1)A(i-1,n)
  j=2 step 1 until n
    $A(i, j) = a_n A(i-1, j-1) + A(0, j)A(i-1, n)$
Set up $n \times n$ *reduced Companion matrix* C
i=1 step 1 until n
  j=1 step 1 until n
    s=0
    k *= 1 step 1 until* m
      If $k - n + j \geq 1$ *then* $s = s + A(k-n+j-1, i)b_k a_n^{m-k}$
      If $i \geq j$ *then* $s = s + a_n^{j+m-n} b_{i-j}$
      $C(i, j) = s$
Remove $\gcd(a_n, b_m)$ *from each element of* C
Remove common numerical factor from each row of C
Remove common numerical factor from each column of C
Calculate the $\binom{n}{2}$ *sub-determinants* H_2
i=1 step 1 until $n-1$
  $j = i+1$ *step 1 until* $n-1$
    $H_2(i, j) = C(i, 1)C(j, 2) - C(i, 2)C(j, 1)$
Remove common numerical factor of the H_2
Remove common factor $a_n^{m-n+1}/\gcd(a_n, b_m)$ *from the* H_2
Remove $\gcd(a_n, b_m)^{min(max(0,m-2n+1),m-n+1)}$ *from the* H_2
Find and remove any other common factors of the H_2
k=3 step 1 until n
  Combine column k *and the* H_{k-1} *to give the* $\binom{n}{k}$ *sub-determinants* H_k
  Remove common numerical factor of the H_k
  Remove common factor $a_n^{m-n+k-1}/\gcd(a_n, b_m)$ *from the* H_k
  Remove $\gcd(a_n, b_m)^{min(max(0,\frac{1}{2}(k-1)(2m+k)-kn),m-n+k-1)}$ *from the* H_k
  Find and remove any other common factors of the H_k
End of algorithm

Example 1c Again using the polynomials of Example 1 we calculate their resultant using the technique of sub-determinants for the Companion matrix. Common factor $(3b-1)(b+3)$ of the leading coefficients is removed from the matrix C; this is equivalent to extracting $(3b-1)^4(b+3)^4$ from the determinant of C. Common numerical factors of 75^9, the constant multiple in a_4^9 (which the determinant must be divided by to give the resultant) and 720, which is a factor of the resultant, are also removed from C. The H_2 with a_4^2 removed have a common factor $2(b+3)(3b-1)^3$, and $20(b+3)(3b-1)(162b+5)\gamma$ is common to the H_3 with a_4^3 removed . Then H_4, before a_4^4 is removed, is a polynomial of degree 96 in b whereas the resultant is of degree 77 in b. Clearly our objective of reducing the size of intermediate expressions has not been achieved for this example.

We find a similar drawback when evaluating the resultant of the polynomials for Example 2. In general the approach based on the Companion matrix leads to expressions that are larger than the required resultant. However, we note that this approach may be advantageous when the degree of f is small compared with that of g, as is demonstrated by Example 3 in the next section.

6. Further examples

We have shown how our approach can be used in conjunction with any of the three methods mentioned for the calculation of resultants. All three methods detect the same factors of the determinant during the course of its calculation, albeit at different stages in the procedures. We use whichever method is found to be the most effective for the case at hand, remembering that the motivation is to minimise the space required rather than the time taken. Although, in general, the Companion matrix approach requires more intermediate storage we find it is the most appropriate method for the first example in this section. Finally we present an example for which our method of sub-determinant calculations based on the Bézout matrix outperforms the resultant procedures within REDUCE and Maple, either in terms of space required or the time taken to perform the calculation.

Example 3 This example serves to illustrate the advantage of the Companion matrix approach when one of the polynomials is of much lower degree than the other and, incidentally, highlights the size of the integer coefficients that occur. It is again based on polynomials which arose in the investigation of the differential system (4) but at the stage when only two variables a, b remain. Note that the first polynomial is homogeneous apart from the second term. If both polynomials were homogeneous and irreducible, then they would be simultaneously zero if and only if $a = b = 0$. In the course of the calculation we need to calculate the resultant of the polynomials Υ, Γ, where

$$\begin{aligned}\Upsilon = {} & 464557320a^8 - 1633110051a^7 - 2037974188a^6b^2 - 21579017205a^5b^3 + \\ & 57307963738a^4b^4 + 189853264347a^3b^5 - 18545338008a^2b^6 + \\ & 30841802109ab^7 + 2818756663338b^8,\end{aligned}$$

$$\begin{aligned}
\Gamma = {} & 825953489617556548896 4a^{22} - 28413830813204734086 1208a^{21}b + \\
& 37084062497766211327 85379a^{20}b^2 - 170171460986303223289 70258a^{19}b^3 - \\
& 915003358427884918628 40869a^{18}b^4 + 153263466636096705453 1403566a^{17}b^5 - \\
& 675006642936129017164 33370686a^{16}b^6 - 123172003842866801056 7454736a^{15}b^7 \\
& +11753964692882643862023 8727628a^{14}b^8 \\
& -335865049700120874407070128512a^{13}b^9 \\
& -4761046107997239245718176 47894a^{12}b^{10} \\
& +413940808416700089360515811 7052a^{11}b^{11} \\
& -410839537113392875034304228 9054a^{10}b^{12} \\
& -183449764473227626299354426 11460a^9b^{13} \\
& +433863320842530578249957359 34408a^8b^{14} \\
& +210614014472470490036132604 82560a^7b^{15} \\
& -137067367296123029900260781 375400a^6b^{16} \\
& +351576168046353631713641063 98536a^5b^{17} \\
& +194641929399070622857185124 352379a^4b^{18} \\
& -636120195022651414756201945 39674a^3b^{19} \\
& -161119480899094799287341029 858469a^2b^{20} \\
& -245553624919122584912713549530ab^{21} \\
& +971969086503136863719637783 79950b^{22}.
\end{aligned}$$

We compared the calculation of $res(\Upsilon, \Gamma, a)$ using various methods. First using the two procedures provided within REDUCE 3.7; the one which is based on the Bézout technique failed with "stack overflow" and the other, which uses sub-resultant calculations, obtained the resultant in $51s$ cpu. Our approach applied to the Bézout matrix fails as all space is exhausted and, because of the limitation on the dimension of arrays, our implementation of the Sylvester matrix approach cannot be used. Of the three routines described in this paper we can only obtain the resultant using the Companion matrix approach. This method is suited to this example because the degree of the variable being eliminated is much smaller in Υ than it is in Γ. The resultant is obtained in $31s$ cpu, which compares favourably with the procedure provided as part of the REDUCE package, particularly as our code was designed to minimise the amount of space required not the time taken.

Example 4 Consider the differential system

$$\begin{aligned}
\dot{x} &= x(x - fy + f - 1)(bx + y + c + d - b), \\
\dot{y} &= y(ax - y - a + 1)(dx + y + c - e).
\end{aligned} \tag{10}$$

This is an example of a Kolmogorov system; such systems are often used by ecologists to model the interaction of two species[12]. There are four points that can be fine foci for system (10) but at most three of these can coexist. Here (10) has been scaled so that one of the fine foci is at $(1,1)$, two other fine foci are $S = (1-m/(a+b), 1-am/(a+b))$ and $T = (1+f(e-m)/(df+1), 1+(e-m)/(df+1))$, where $m = c+d+1$. We can transform (10) so that it is of the form (1) with origin at each of these points in turn. For each transformed system we must have $\lambda = 0$; for the system with origin at $(1,1)$ this means $e = 0$ and for the other two systems this condition gives expressions for the variables a, b. When considering the focal value η_4 for the system with origin at S we need to investigate the possibility of $P_1 = P_2 = 0$. Here P_1, a polynomial with 389 terms, is degree 11 in each of the three remaining variables d, f, m and P_2, with 263 terms, is degree 9 in d, m, and degree 10 in f. Very few of the integer coefficients are greater than 1000. See the Appendix for details of P_1, P_2. Attempting to calculate $res(P_1, P_2, m)$ using the REDUCE procedure based on the Bézout technique, or using the Maple resultant procedure, leads to all available space being exhausted. The resultant can be computed using the REDUCE procedure which uses sub-resultant calculations giving $res(P_1, P_2, m) = 512d^{23}f^{48}\Phi$, where Φ is a polynomial of degrees 56 in d, 138 in f. This calculation took over 33 hours cpu time, of which more than 67% was taken by garbage collection. Using our procedure based on the Bézout matrix we find, during the calculation of the determinant, that $256d^{14}f^{37}(f-1)^{58}(f+1)^2(df+1)^{32}$ is a factor of the resultant with cofactor $2d^9f^{11}\Theta$, where Θ is a polynomial of degrees 24 in d, 46 in f. This calculation required approximately 13.5 hours cpu time with less than 3% of the time being spent in garbage collection. Also the factorisation of Θ, to obtain the resultant in its simplest form, is trivial compared with that of Φ. There is a clear advantage in using our technique when the resultant has many simple factors to high multiplicity.

We have

$$\begin{aligned} res(P_1, P_2, m) \quad = \quad & 512d^{23}(d-1)f^{48}(f-1)^{76}(f+1)^4(d+f)(df+1)^{49} \times \\ & (d-f^2+f+1)^2(d^2f^2-2df^2+3df+f^2-2f+1) \times \\ & (df^2+2df+d-2f^2+4f+2). \end{aligned}$$

This result leads to the conclusion that the three coexisting fine foci can each be of order at most one.

7. Conclusion

We have presented three different approaches to the calculation of the resultant of two multivariate polynomials and have compared their performance with reference to examples. The technique we use differs from other methods in that some factors of the resultant are removed during the course of its calculation. This reduces the amount of storage space required and the stack size needed for individual expressions. Obtaining all the factors of the resultant in their simplest terms is expedited as the factorisation of one very large expression is replaced by several factorisations of simpler functions.

The approach based on the Sylvester matrix seems to offer the least potential. It requires the calculation of by far the most sub-determinants, although each sub-determinant in the early stages contains fewer terms than those for the other methods. It also has the added complication of the need to determine which sub-determinants to calculate. For efficiency we store the parameters $h_{k,i}$ in an array rather than a list in our implementation using REDUCE. There is a maximum array size set within the REDUCE system we use and we soon reach this limit. Currently we cannot consider polynomials of degree greater than nine in the variable being eliminated using this software. In the Sylvester matrix approach no common factors other than factors of the leading coefficients can arise in the H_k, for $k \leq m$, so if $m-n$ is small there is little opportunity to detect more interesting factors of the resultant at an early stage in the calculation of the H_k. Conversely, when using the Bézout matrix technique with $m-n \leq 2$ it is possible that such factors will arise from H_3 onwards.

In certain examples expressions generated during the Companion matrix approach are even larger than the resultant being sought, but we have given one instance, in Example 3, where this method outperforms the other two and those procedures which are provided as part of the REDUCE 3.7 package. When the degree of the variable being eliminated is much lower in one of the polynomials than the other the Companion matrix approach can be advantageous. In general the Bézout matrix algorithm is the most useful. Example 4, where the resultant has many simple factors of high multiplicity, clearly demonstrates the advantage of removing factors of the determinant during its calculation.

The extent of any advantage given by the removal of factors during the course of the calculation is very much problem dependent. We have found that our method, based on the calculation of sub-determinants and the removal of factors of the resultant as they arise, offers advantages over other available procedures for the resultants we need to calculate. Indeed we have been able to complete some investigations that were otherwise intractable.

Appendix

$$
\begin{aligned}
P_1 = \quad & d^{11}f^{11} - 6d^{10}f^{11}m - 4d^{10}f^{10}m + 11d^{10}f^{10} + 16d^9f^{11}m^2 - 2d^9f^{11}m + \\
& 24d^9f^{10}m^2 - 54d^9f^{10}m + 4d^9f^9m^2 - 44d^9f^9m + 55d^9f^9 - 26d^8f^{11}m^3 \\
& +12d^8f^{11}m^2 - 2d^8f^{11}m - 62d^8f^{10}m^3 + 121d^8f^{10}m^2 - 12d^8f^{10}m - \\
& 28d^8f^9m^3 + 208d^8f^9m^2 - 222d^8f^9m + 4d^8f^8m^3 + 55d^8f^8m^2 - \\
& 214d^8f^8m + 165d^8f^8 + 30d^7f^{11}m^4 - 30d^7f^{11}m^3 + 9d^7f^{11}m^2 + \\
& 90d^7f^{10}m^4 - 180d^7f^{10}m^3 + 74d^7f^{10}m^2 - 16d^7f^{10}m + 86d^7f^9m^4 - \\
& 428d^7f^9m^3 + 405d^7f^9m^2 - 24d^7f^9m - 16d^7f^8m^4 - 254d^7f^8m^3 + \\
& 804d^7f^8m^2 - 552d^7f^8m - 8d^7f^7m^4 - 4d^7f^7m^3 + 292d^7f^7m^2 - \\
& 608d^7f^7m + 330d^7f^7 - 26d^6f^{11}m^5 + 40d^6f^{11}m^4 - 16d^6f^{11}m^3 -
\end{aligned}
$$

$80d^6f^{10}m^5 + 205d^6f^{10}m^4 - 176d^6f^{10}m^3 + 59d^6f^{10}m^2 - 150d^6f^9m^5 + 538d^6f^9m^4 - 542d^6f^9m^3 + 198d^6f^9m^2 - 56d^6f^9m + 16d^6f^8m^5 + 507d^6f^8m^4 - 1282d^6f^8m^3 + 767d^6f^8m^2 + 42d^6f^7m^5 + 20d^6f^7m^4 - 976d^6f^7m^3 + 1836d^6f^7m^2 - 924d^6f^7m + 2d^6f^6m^5 - 36d^6f^6m^4 - 144d^6f^6m^3 + 836d^6f^6m^2 - 1120d^6f^6m + 462d^6f^6 + 16d^5f^{11}m^6 - 30d^5f^{11}m^5 + 14d^5f^{11}m^4 + 44d^5f^{10}m^6 - 166d^5f^{10}m^5 + 204d^5f^{10}m^4 - 82d^5f^{10}m^3 + 160d^5f^9m^6 - 492d^5f^9m^5 + 633d^5f^9m^4 - 466d^5f^9m^3 + 165d^5f^9m^2 + 20d^5f^8m^6 - 40d^5f^7m^5 + 1323d^5f^7m^4 - 2136d^5f^7m^3 + 857d^5f^7m^2 + 84d^5f^7m - 14d^5f^6m^6 + 136d^5f^6m^5 + 354d^5f^6m^4 - 2140d^5f^6m^3 + 2756d^5f^6m^2 - 1092d^5f^6m + 2d^5f^5m^6 + 24d^5f^5m^5 - 8d^5f^5m^4 - 540d^5f^5m^3 + 1460d^5f^5m^2 - 1400d^5f^5m + 462d^5f^5 - 6d^4f^{11}m^7 + 12d^4f^{11}m^6 - 6d^4f^{11}m^5 - 14d^4f^{10}m^7 + 79d^4f^{10}m^6 - 116d^4f^{10}m^5 + 51d^4f^{10}m^4 - 104d^4f^9m^7 + 332d^4f^9m^6 - 522d^4f^9m^5 + 464d^4f^9m^4 - 170d^4f^9m^3 - 60d^4f^8m^7 + 517d^4f^8m^6 - 1084d^4f^8m^5 + 1112d^4f^8m^4 - 740d^4f^8m^3 + 255d^4f^8m^2 + 90d^4f^7m^7 + 52d^4f^7m^6 - 1064d^4f^7m^5 + 1642d^4f^7m^4 - 890d^4f^7m^3 + 310d^4f^7m^2 - 140d^4f^7m + 40d^4f^6m^7 - 208d^4f^6m^6 - 288d^4f^6m^5 + 1877d^4f^6m^4 - 2090d^4f^6m^3 + 501d^4f^6m^2 + 168d^4f^6m - 10d^4f^5m^7 - 78d^4f^5m^6 + 112d^4f^5m^5 + 1046d^4f^5m^4 - 2990d^4f^5m^3 + 2844d^4f^5m^2 - 924d^4f^5m - 6d^4f^4m^6 + 28d^4f^4m^5 + 178d^4f^4m^4 - 960d^4f^4m^3 + 1634d^4f^4m^2 - 1204d^4f^4m + 330d^4f^4 + d^3f^{11}m^8 - 2d^3f^{11}m^7 + d^3f^{11}m^6 + 2d^3f^{10}m^8 - 16d^3f^{10}m^7 + 26d^3f^{10}m^6 - 12d^3f^{10}m^5 + 38d^3f^9m^8 - 140d^3f^9m^7 + 235d^3f^9m^6 - 202d^3f^9m^5 + 69d^3f^9m^4 + 56d^3f^8m^8 - 318d^3f^8m^7 + 694d^3f^8m^6 - 838d^3f^8m^5 + 586d^3f^8m^4 - 180d^3f^8m^3 - 40d^3f^7m^8 - 68d^3f^7m^7 + 664d^3f^7m^6 - 1264d^3f^7m^5 + 1243d^3f^7m^4 - 770d^3f^7m^3 + 235d^3f^7m^2 - 60d^3f^6m^8 + 192d^3f^6m^7 + 120d^3f^6m^6 - 944d^3f^6m^5 + 1058d^3f^6m^4 - 476d^3f^6m^3 + 222d^3f^6m^2 - 112d^3f^6m + 20d^3f^5m^8 + 72d^3f^5m^7 - 220d^3f^5m^6 - 422d^3f^5m^5 + 1483d^3f^5m^4 - 1132d^3f^5m^3 + 31d^3f^5m^2 + 168d^3f^5m + 24d^3f^4m^7 - 114d^3f^4m^6 - 202d^3f^4m^5 + 1574d^3f^4m^4 - 2758d^3f^4m^3 + 2028d^3f^4m^2 - 552d^3f^4m - 6d^3f^3m^6 - 36d^3f^3m^5 + 357d^3f^3m^4 - 956d^3f^3m^3 + 1180d^3f^3m^2 - 704d^3f^3m + 165d^3f^3 - 6d^2f^9m^9 + 26d^2f^9m^8 - 42d^2f^9m^7 + 30d^2f^9m^6 - 8d^2f^9m^5 - 24d^2f^8m^9 + 121d^2f^8m^8 - 260d^2f^8m^7 + 294d^2f^8m^6 - 172d^2f^8m^5 + 41d^2f^8m^4 - 2d^2f^7m^9 + 76d^2f^7m^8 - 330d^2f^7m^7 + 654d^2f^7m^6 - 712d^2f^7m^5 + 414d^2f^7m^4 - 100d^2f^7m^3 + 50d^2f^6m^9 -$

$148d^2f^6m^8 - 12d^2f^6m^7 + 557d^2f^6m^6 - 998d^2f^6m^5 + 942d^2f^6m^4 -$
$520d^2f^6m^3 + 129d^2f^6m^2 - 20d^2f^5m^9 + 12d^2f^5m^8 + 68d^2f^5m^7 +$
$8d^2f^5m^6 - 198d^2f^5m^5 + 162d^2f^5m^4 - 90d^2f^5m^3 + 114d^2f^5m^2 -$
$56d^2f^5m - 36d^2f^4m^8 + 174d^2f^4m^7 - 179d^2f^4m^6 - 280d^2f^4m^5 +$
$618d^2f^4m^4 - 238d^2f^4m^3 - 155d^2f^4m^2 + 96d^2f^4m + 18d^2f^3m^7 +$
$30d^2f^3m^6 - 496d^2f^3m^5 + 1350d^2f^3m^4 - 1644d^2f^3m^3 + 964d^2f^3m^2 -$
$222d^2f^3m + 6d^2f^2m^6 - 76d^2f^2m^5 + 295d^2f^2m^4 - 544d^2f^2m^3 +$
$532d^2f^2m^2 - 268d^2f^2m + 55d^2f^2 + 4df^8m^{10} - 20df^8m^9 + 40df^8m^8 -$
$40df^8m^7 + 20df^8m^6 - 4df^8m^5 + 8df^7m^10 - 48df^7m^9 + 121df^7m^8 -$
$64df^7m^7 + 126df^7m^6 - 52df^7m^5 + 9df^7m^4 - 22df^6m^{10} + 88df^6m^9 -$
$90df^6m^8 - 106df^6m^7 + 334df^6m^6 - 324df^6m^5 + 146df^6m^4 - 26df^6m^3$
$+10df^5m^{10} - 48df^5m^9 + 116df^5m^8 - 234df^5m^7 + 425df^5m^6 -$
$560df^5m^5 + 458df^5m^4 - 206df^5m^3 + 39df^5m^2 + 24df^4m^9 - 118df^4m^8$
$+252df^4m^7 - 334df^4m^6 + 320df^4m^5 - 194df^4m^4 + 28df^4m^3 + 38df^4m^2$
$-16df^4m - 18df^3m^8 + 48df^3m^7 + 15df^3m^6 - 150df^3m^5 + 120df^3m^4 +$
$48df^3m^3 - 93df^3m^2 + 30df^3m - 12df^2m^7 + 108df^2m^6 - 366df^2m^5 +$
$624df^2m^4 - 576df^2m^3 + 276df^2m^2 - 54df^2m + 6dfm^6 - 40dfm^5 +$
$111dfm^4 - 164dfm^3 + 136dfm^2 - 60dfm + 11df - 2f^7m^{11} + 12f^7m^{10}$
$-30f^7m^9 + 40f^7m^8 - 30f^7m^7 + 12f^7m^6 - 2f^7m^5 + 4f^6m^{11} -$
$24f^6m^{10} + 60f^6m^9 - 80f^6m^8 + 60f^6m^7 - 24f^6m^6 + 4f^6m^5 - 2f^5m^{11}$
$+18f^5m^{10} - 68f^5m^9 + 142f^5m^8 - 180f^5m^7 + 142f^5m^6 - 68f^5m^5 +$
$18f^5m^4 - 2f^5m^3 - 6f^4m^{10} + 30f^4m^9 - 49f^4m^8 + 105f^4m^6 -$
$154f^4m^5 + 105f^4m^4 - 36f^4m^3 + 5f^4m^2 + 6f^3m^9 - 42f^3m^8 +$
$124f^3m^7 - 198f^3m^6 + 180f^3m^5 - 86f^3m^4 + 12f^3m^3 + 6f^3m^2 -$
$2f^3m + 6f^2m^8 - 32f^2m^7 + 66f^2m^6 - 60f^2m^5 + 10f^2m^4 +$
$24f^2m^3 - 18f^2m^2 + 4f^2m - 6fm^7 + 36fm^6 - 90fm^5 + 120fm^4 -$
$90fm^3 + 36fm^2 - 6fm + m^6 - 6m^5 + 15m^4 - 20m^3 + 15m^2 - 6m + 1,$

$P_2 =$ $d^9f^9 + 2d^8f^{10}m - 2d^8f^{10} - 7d^8f^9m + 4d^8f^9 - 4d^8f^8m + 7d^8f^8 -$
$10d^7f^{10}m^2 + 10d^7f^{10}m + 17d^7f^9m^2 + 2d^7f^9m - 16d^7f^9 + 20d^7f^8m^2$
$-57d^7f^8m + 32d^7f^8 + 8d^7f^7m^2 - 27d^7f^7m + 20d^7f^7 + 20d^6f^{10}m^3$
$-20d^6f^{10}m^2 - 17d^6f^9m^3 - 50d^6f^9m^2 + 66d^6f^9m - 38d^6f^8m^3 +$
$126d^6f^8m^2 - 30d^6f^8m - 56d^6f^8 - 36d^6f^7m^3 + 138d^6f^7m^2 -$
$215d^6f^7m + 112d^6f^7 - 6d^6f^6m^3 + 51d^6f^6m^2 - 73d^6f^6m + 28d^6f^6 -$
$20d^5f^{10}m^4 + 20d^5f^{10}m^3 + 3d^5f^9m^4 + 100d^5f^9m^3 - 103d^5f^9m^2 +$

$32d^5f^8m^4 - 90d^5f^8m^3 - 128d^5f^8m^2 + 186d^5f^8m + 64d^5f^7m^4 -$
$253d^5f^7m^3 + 411d^5f^7m^2 - 110d^5f^7m - 112d^5f^7 + 26d^5f^6m^4 -$
$195d^5f^6m^3 + 430d^5f^6m^2 - 485d^5f^6m + 224d^5f^6 - 44d^5f^5m^3 +$
$125d^5f^5m^2 - 95d^5f^5m + 14d^5f^5 + 10d^4f^{10}m^5 - 10d^4f^{10}m^4 +$
$7d^4f^9m^5 - 80d^4f^9m^4 + 73d^4f^9m^3 - 8d^4f^8m^5 - 21d^4f^8m^4 +$
$244d^4f^8m^3 - 215d^4f^8m^2 - 56d^4f^7m^5 + 200d^4f^7m^4 - 194d^4f^7m^3$
$-240d^4f^7m^2 + 290d^4f^7m - 44d^4f^6m^5 + 286d^4f^6m^4 - 692d^4f^6m^3 +$
$760d^4f^6m^2 - 170d^4f^6m - 140d^4f^6 - 2d^4f^5m^5 + 132d^4f^5m^4 -$
$485d^4f^5m^3 + 780d^4f^5m^2 - 705d^4f^5m + 280d^4f^5 + 2d^4f^4m^5 +$
$18d^4f^4m^4 - 101d^4f^4m^3 + 140d^4f^4m^2 - 45d^4f^4m - 14d^4f^4 -$
$2d^3f^{10}m^6 + 2d^3f^{10}m^5 - 5d^3f^9m^6 + 26d^3f^9m^5 - 21d^3f^9m^4 -$
$4d^3f^8m^6 + 51d^3f^8m^5 - 144d^3f^8m^4 + 97d^3f^8m^3 + 24d^3f^7m^6 -$
$57d^3f^7m^5 - 88d^3f^7m^4 + 351d^3f^7m^3 - 230d^3f^7m^2 + 36d^3f^6m^6 -$
$198d^3f^6m^5 + 388d^3f^6m^4 - 166d^3f^6m^3 - 330d^3f^6m^2 + 270d^3f^6m +$
$8d^3f^5m^6 - 144d^3f^5m^5 + 579d^3f^5m^4 - 1048d^3f^5m^3 + 855d^3f^5m^2 -$
$138d^3f^5m - 112d^3f^5 - 8d^3f^4m^6 - 40d^3f^4m^5 + 296d^3f^4m^4 -$
$685d^3f^4m^3 + 880d^3f^4m^2 - 667d^3f^4m + 224d^3f^4 - 4d^3f^3m^5 +$
$40d^3f^3m^4 - 89d^3f^3m^3 + 50d^3f^3m^2 + 31d^3f^3m - 28d^3f^3 + d^2f^9m^7 -$
$2d^2f^9m^6 + d^2f^9m^5 + 2d^2f^8m^7 - 16d^2f^8m^6 + 26d^2f^8m^5 - 12d^2f^8m^4$
$-4d^2f^7m^7 - 2d^2f^7m^6 + 69d^2f^7m^5 - 116d^2f^7m^4 + 53d^2f^7m^3$
$-14d^2f^6m^7 + 63d^2f^6m^6 - 55d^2f^6m^5 - 153d^2f^6m^4 + 289d^2f^6m^3 -$
$130d^2f^6m^2 - 12d^2f^5m^7 + 80d^2f^5m^6 - 233d^2f^5m^5 + 284d^2f^5m^4 +$
$21d^2f^5m^3 - 290d^2f^5m^2 + 150d^2f^5m + 12d^2f^4m^7 + 12d^2f^4m^6 -$
$246d^2f^4m^5 + 706d^2f^4m^4 - 952d^2f^4m^3 + 582d^2f^4m^2 - 58d^2f^4m -$
$56d^2f^4 + 12d^2f^3m^6 - 102d^2f^3m^5 + 320d^2f^3m^4 - 555d^2f^3m^3 +$
$610d^2f^3m^2 - 397d^2f^3m + 112d^2f^3 - 6d^2f^2m^5 + 21d^2f^2m^4 -$
$11d^2f^2m^3 - 37d^2f^2m^2 + 53d^2f^2m - 20d^2f^2 + df^7m^7 - 3df^7m^6 +$
$3df^7m^5 - df^7m^4 + 2df^6m^8 - 7df^6m^7 + 2df^6m^6 + 16df^6m^5 - 20df^6m^4$
$+7df^6m^3 + 8df^5m^8 - 36df^5m^7 + 44df^5m^6 + 39df^5m^5 - 141df^5m^4 +$
$121df^5m^3 - 35df^5m^2 - 8df^4m^8 + 24df^4m^7 + 8df^4m^6 - 86df^4m^5 +$
$44df^4m^4 + 112df^4m^3 - 140df^4m^2 + 46df^4m - 12df^3m^7 + 84df^3m^6 -$
$275df^3m^5 + 499df^3m^4 - 491df^3m^3 + 221df^3m^2 - 10df^3m - 16df^3 +$
$12df^2m^6 - 61df^2m^5 + 150df^2m^4 - 236df^2m^3 + 238df^2m^2 - 135df^2m +$
$32df^2 - 6dfm^4 + 25dfm^3 - 39dfm^2 + 27dfm - 7df - 2f^5m^9 + 12f^5m^8$
$-30f^5m^7 + 40f^5m^6 - 30f^5m^5 + 12f^5m^4 - 2f^5m^3 + 2f^4m^9 - 14f^4m^8$

$$+43f^4m^7 - 75f^4m^6 + 80f^4m^5 - 52f^4m^4 + 19f^4m^3 - 3f^4m^2 + 4f^3m^8$$
$$-22f^3m^7 + 44f^3m^6 - 30f^3m^5 - 20f^3m^4 + 46f^3m^3 - 28f^3m^2 + 6f^3m$$
$$-6f^2m^7 + 40f^2m^6 - 108f^2m^5 + 150f^2m^4 - 110f^2m^3 + 36f^2m^2 - 2f^2$$
$$-4fm^5 + 20fm^4 - 40fm^3 + 40fm^2 - 20fm + 4f + m^5 - 5m^4 + 10m^3$$
$$-10m^2 + 5m - 1.$$

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